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Viscous dissipation in external natural convection flows

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The effects of viscous dissipation are considered for external natural convection flow over a surface. A class of similar boundary-layer solutions is given and numerical results are presented for a wide range of the dissipation and Prandtl numbers. Several general aspects of similarity conditions for flow over surfaces and in convection plumes are discussed and their special characteristics considered. The general equations including the dissipation effect are given for the non-similar power law surface condition.

Introduction

Many natural convection processes encountered are not sufficiently vigorous to result in a viscous dissipation effect (i.e. a thermalization of energy through the mechanism of viscous stresses) which is appreciable compared to other energy flows in the convection region. However, it is clear (see Gebhart 1962) that in natural convection flow fields of extreme size, or extremely low temperatures, or in high gravity the viscous generation of heat will affect the flow. In that paper external boundary-layer flow over a vertical flat surface was considered. The dissipation effect is non-similar for the most common boundary conditions and a dissipation parameter, $\epsilon(x) = g\beta x/c_p$,† arose in a perturbation analysis of the uniform temperature and the uniform heat flux surface conditions. The effect was calculated for a Prandtl number (σ) range from 10^{-2} to 10^4, and was found to increase over that range.

Other studies of this effect and the application of external two-dimensional flow results to internal flows are discussed in Gebhart (1962). Since that time one other study has come to the writers' attention. Apparently Roy (1968, personal communication) has found an asymptotic solution for large Prandtl number for external flow on an isothermal surface, using a double boundary-layer concept.

The present paper shows that a similarity solution exists for the external flow case over an infinite flat vertical surface, in an extensive medium at uniform temperature t_∞ having an exponential variation of surface temperature t_0, i.e. $t_0 - t_\infty = Me^{mx}$. This similarity means that one may calculate the convection field, to the accuracy of the conventional boundary-layer equations, in terms of two parameters, the Prandtl number σ and a dissipation parameter $g\beta/mc_p$, where $1/m$ is the e-folding distance. The results of numerical calculations are presented

† g is gravity, β is the volumetric coefficient of thermal expansion, x is the distance from the leading edge of an infinite vertical surface, and c_p is the applicable specific heat.

for a wide range of these parameters. The special characteristics and peculiar features of the convection field implied by an exponential variation are considered in appendix A.

Analysis

The general equations of fluid motion for two-dimensional flow in Cartesian co-ordinates may be reduced to the simpler boundary-region form, as found in the experimental observations of Schmidt & Beckmann (1930) and as indicated by an order of magnitude analysis. The first-order treatment of variable density, the Boussinesq approximation, is to include the effect only in the body-force–pressure-gradient terms and as the first term of an expansion around ρ_∞. The equations for steady flow over an infinitely wide flat surface, parallel to the body force, with uniform fluid kinematic viscosity ν and thermal conductivity k and including viscous dissipation, are:

$$\nabla . \mathbf{w} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = g\beta(t-t_\infty)+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 t}{\partial y^2}+\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2. \tag{3}$$

Letting $u(x,y) \equiv \nu\psi_y, v(x,y) \equiv -\nu\psi_x, d(x) \equiv t_0-t_\infty$ and $\phi \equiv (t-t_\infty)/(t_0-t_\infty)$ we obtain

$$\psi_y\,\psi_{yx}-\psi_x\psi_{yy} = \frac{g\beta d\phi}{\nu^2}+\psi_{yyy}, \tag{4}$$

$$\psi_y\phi_x d+\psi_y\phi d_x-\psi_x\phi_y d = \frac{1}{\sigma}\phi_{yy}d+\frac{\nu^2}{c_p}\psi_{yy}^2, \tag{5}$$

where for $t_0 > t_\infty$, x is measured positive upward.

The retention of the viscous dissipation term in (5) is not justified for a gas if the pressure term and the pressure effect on density are omitted. Therefore, this analysis applies to liquids and c_p is the appropriate specific heat.

Assuming a similarity variable and a stream function of the form:

$$\eta = yb(x), \tag{6}$$

$$\psi(x,y) = c(x)f(\eta), \tag{7}$$

we find the conditions that

$$\phi(x,y) = \phi(\eta). \tag{8}$$

The equations become

$$f'''+\frac{\beta g d}{\nu^2 cb^3}\phi+\frac{c_x}{b}ff''-\left(\frac{c_x}{b}+\frac{cb_x}{b^2}\right)(f')^2 = 0, \tag{9}$$

$$\frac{1}{\sigma}\phi''+\frac{c_x}{b}f\phi'-\frac{d_x c}{bd}f'\phi+\frac{\nu^2c^2b^2}{dc_p}(f'')^2 = 0. \tag{10}$$

The last term in (10) is the viscous dissipation effect. It is well known that when this term is neglected, similarity solutions exist for a power law $[d(x) = Nx^n]$

and for exponential $[d(x) = Me^{mx}]$ surface (at $y = 0$) temperature distributions. The equations, in this form, are as follows:

Power law:

$$\left.\begin{aligned} & f''' + \phi + (m+3)ff'' - (2M+2)(f')^2 = 0, \\ & \phi'' + \sigma[(n+3)f\phi' - 4Mf'\phi] = 0, \end{aligned}\right\} \tag{11}$$

$$d(x) = Nx^n,$$
$$c(x) = 4[\tfrac{1}{4}Gr_x]^{\frac{1}{4}},$$
$$b(x) = 1/x\,[\tfrac{1}{4}Gr_x]^{\frac{1}{4}},$$
$$Gr_x \equiv g\beta x^3 N x^n/\nu^2.$$

Exponential:

$$\left.\begin{aligned} & f''' + \phi + ff'' - 2(f')^2 = 0, \\ & \phi'' + \sigma(f\phi' - 4f'\phi) = 0, \end{aligned}\right\} \tag{12}$$

$$d(x) = M\,e^{mx},$$
$$c(x) = 4[\tfrac{1}{4}Gr_m]^{\frac{1}{4}},$$
$$b(x) = m[\tfrac{1}{4}Gr_m]^{\frac{1}{4}},$$
$$Gr_m \equiv g\beta M\,e^{mx}/m^3\nu^2.$$

Therefore, when viscous dissipation is neglected, f and ϕ are functions of η, σ, and the exponent n for the power law case, and are functions of η and σ for the exponential case.

To find physically meaningful values of n and m for these cases we consider the local heat flux from the surface. For the flat surface the heat flux is given by

$$q''(x) = -k\left.\frac{\partial t}{\partial y}\right|_{y=0} = k[-\phi'(0)]\,b(x)\,d(x), \tag{13}$$

where heat flow in the direction of increasing y, i.e. to the fluid, is taken to be positive.

The proper $b(x)$ and $d(x)$ give the following heat flux for the power law and for the exponential cases, respectively.

$$q''(x) = \left\{k[-\phi'(0)]\,N\left[\frac{g\beta N}{4\nu^2}\right]^{\frac{1}{4}}\right\} x^{\frac{1}{4}(5n-1)}, \tag{14}$$

$$q''(x) = \left\{k[-\phi'(0)]\,M\left[\frac{g\beta M}{4\nu^2}\right]^{\frac{1}{4}}\right\} m^{\frac{1}{4}}\,e^{\frac{5}{4}mx}, \tag{15}$$

which may be written as,

$$q''(x) = K_1 x^{\frac{1}{4}(5n-1)}, \tag{16}$$

$$q''(x) = K_2 m^{\frac{1}{4}} e^{\frac{5}{4}mx}, \tag{17}$$

where K_1 and K_2 are *positive* constants for $t_0 > t_\infty$. The total heat transfer from zero to x per unit width, is given by

$$q(x) = \int_0^x q''(x)\,dx. \tag{18}$$

For the power law case (for $n \neq -\frac{3}{5}$) and for the exponential case respectively,

$$q(x) = K_1\left(\frac{4}{5n+3}\right) x^{\frac{1}{4}(5n+3)}, \tag{19}$$

$$q(x) = \tfrac{4}{5}K_2 m^{-\frac{3}{4}}[e^{\frac{5}{4}mx} - 1]. \tag{20}$$

Since $q(x)$ must be positive it follows that

$$n > -\tfrac{3}{5}, \tag{21}$$

$$m > 0. \tag{22}$$

For $t_0 < t_\infty$ the same conclusion is reached.

Further, we will examine conditions necessary for meaningful values of m and n for another common geometry. For the case of a plume above a uniform line heat source, or over any plane heat source, the energy in the plume a distance x above its source, or at the plate's beginning, per unit width is:

$$q(x) = \int_{-\infty}^{+\infty} u\rho c_p (t - t_\infty)\, dy = 2\nu\rho c_p c(x)\, d(x) \int_0^\infty f'\phi\, d\eta. \tag{23}$$

Again, introducing the proper $c(x)$ and $d(x)$, the plume energy content, for the power law and the exponential plume mid-plane temperature variation (t_0) are:

$$q(x) = \frac{8\rho c_p \nu}{\sqrt{2}} \left[\frac{g\beta N}{\nu^2}\right]^{\frac{1}{4}} \left[\int_0^\infty f'\phi\, d\eta\right] x^{\frac{1}{4}(5n+3)}, \tag{24}$$

$$q(x) = \frac{8\rho c_p \nu}{\sqrt{2}} \left[\frac{g\beta M}{\nu^2}\right]^{\frac{1}{4}} \left[\int_0^\infty f'\phi\, d\eta\right] m^{-\frac{3}{4}}\, M e^{\frac{5}{4}mx}. \tag{25}$$

From (24) it is seen that, for the power law case, $n = -\frac{3}{5}$ describes a line source, i.e. no x dependence of $q(x)$. The $n = -\frac{3}{5}$ is not an important case for flow over a surface, the result is an infinite energy flux in the vicinity of the leading edge with the rest of the surface adiabatic.

Equation (25) shows that the exponential case cannot describe a plume arising entirely from a line source because $q(x)$ is variable with x for all non-zero values of m.

Hence, realistic local heat flux requires

$$n > -\tfrac{3}{5}, \tag{26}$$

$$m > 0, \tag{27}$$

for a flat plate, and

$$n = -\tfrac{3}{5} \tag{28}$$

for a uniform line heat source.

When the effect of viscous dissipation is included, similarity is spoiled for the power law case by an x dependence of the coefficient of the viscous dissipation term in (10):

$$\frac{\nu^2 c^2 b^2}{c_p d} = 4\frac{g\beta x}{c_p} = 4\epsilon(x). \tag{29}$$

This is the perturbation parameter used by Gebhart (1962).

For the exponential case

$$\frac{\nu^2 c^2 b^2}{c_p d} = 4\frac{g\beta}{M c_p}, \tag{30}$$

and we have similarity with this new, additional parameter.

Hence for the exponential case, including the effect of viscous dissipation, we have the following equations:

$$\left.\begin{aligned} f''' + \phi + ff'' - 2(f')^2 &= 0,\\ \phi'' + \sigma\left[f\phi' - 4f'\phi + 4\frac{g\beta}{mc_p}(f'')^2\right] &= 0.\end{aligned}\right\} \tag{31}$$

Note that m now appears and f and ϕ are now functions of η, σ, and a viscous dissipation parameter, $4g\beta/mc_p$. This new parameter is proportional to the kinetic energy of the flow divided by the heat transferred to the fluid, and is similar to the dissipation parameter, $\epsilon(x)$, found in the perturbation analysis for the power law case. The e-folding parameter $1/m$ replaces x found in that analysis.

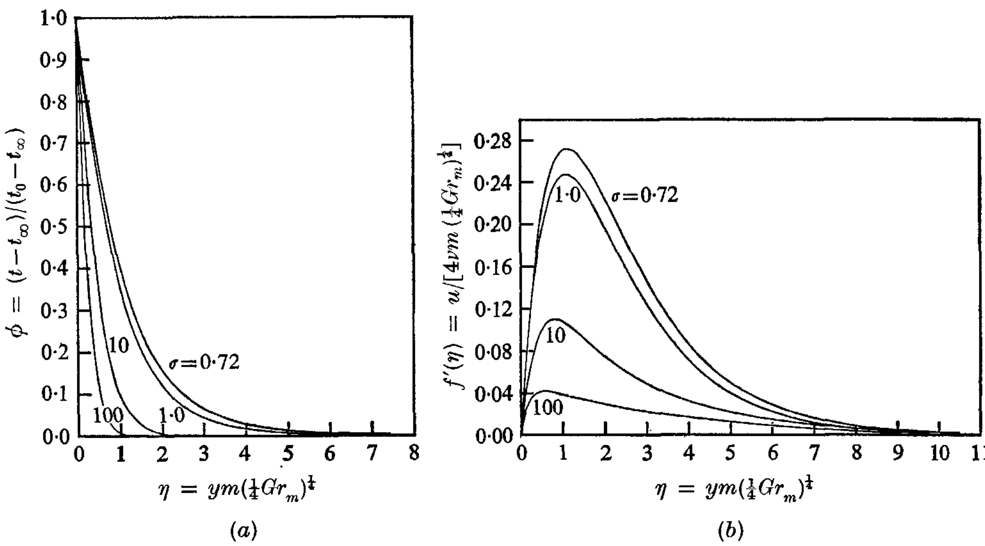

FIGURE 1. Non-dissipative velocity and temperature distributions for various Prandtl numbers. (*a*) Temperature distribution. (*b*) Velocity distribution.

The resulting equations from the previous perturbation analysis have been solved for various Prandtl numbers for both the isothermal and the uniform flux surface boundary conditions. In appendix B the equations are presented for any surface power law temperature distribution. Note that $n = 0$, and $n = \frac{1}{5}$ correspond to isothermal and uniform flux surfaces, respectively; and that $n = -\frac{3}{5}$ corresponds to a plume rising from a uniform line source.

Equations (31) were numerically integrated for the vertical surface case, with the boundary conditions

$$\left.\begin{aligned} &\text{at} \quad \eta = 0\colon \quad f = 0, \quad f' = 0, \quad \phi = 1;\\ &\text{as} \quad \eta \to \infty\colon \quad f' \to 0, \quad \phi \to 0, \end{aligned}\right\} \tag{32}$$

for Prandtl numbers of 0·72, 1, 10 and 100, and for a range of the dissipation parameter, $4g\beta/mc_p$ from $0 \to 2{\cdot}0$, for each Prandtl number. Calculation consisted of a shooting method, using a predictor–corrector to integrate, and the technique described by Nachtsheim & Swigert (1965) to correct the first choices of initial values. Our results for negligible dissipation agree with those of Sparrow & Gregg (1958) for the two Prandtl number values they considered.

Velocity and temperature distributions are shown in figure 1 for a zero dissipation effect, for the various Prandtl numbers. Heat transfer, $q''(x)$, and local Nusselt numbers are known from the ϕ distribution as follows:

$$q''(x) = -k\frac{\partial t}{\partial y}\bigg|_{y=0} = k(t_0 - t_\infty)\, b(x)\,[-\phi'(0)],$$

$$\frac{q''(x)}{t_0 - t_\infty}\frac{x}{k} = \frac{h_x x}{k} = Nu_x = [-\phi'(0)]\, x b(x) = [-\phi'(0)]\, mx\left(\frac{Gr_m}{4}\right)^{\frac{1}{4}}.$$

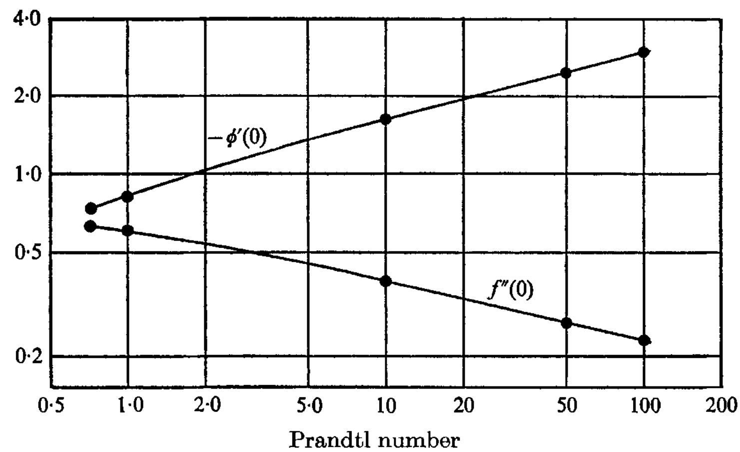

FIGURE 2. Heat transfer and drag parameters in the absence of a viscous dissipation effect.

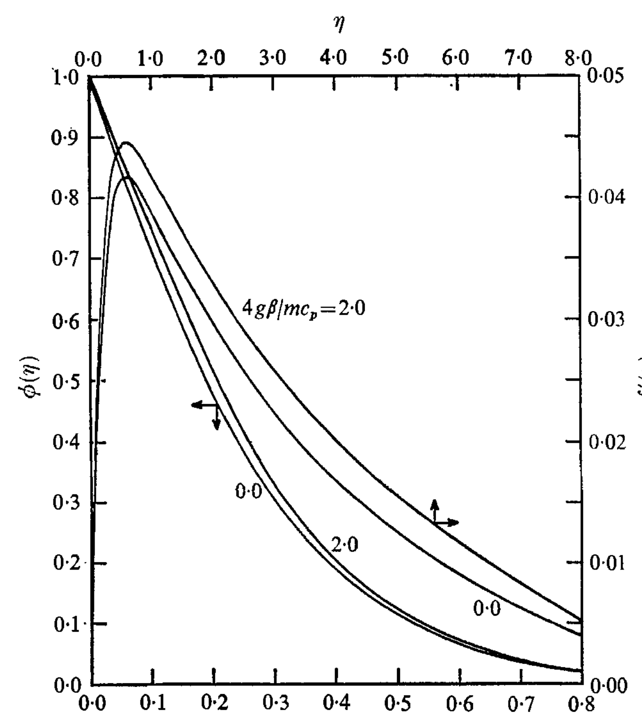

FIGURE 3. Effect of viscous dissipation on the velocity and temperature profiles for $\sigma = 100$.

The viscous stress $\tau(x)$ and local drag coefficient $C_d(x)$, defined on a 'convection velocity' $U_c = \nu c(x)\, b(x) f'_{\max} \propto \nu c(x)\, b(x)$, are known from $f(\eta)$.

$$\tau(x) = \mu \frac{\partial u}{\partial y} = \mu\nu c(x)\, b^2(x) f''(0) = 4\mu\nu m^2 f''(0)\,(\tfrac{1}{4}Gr_m)^{\frac{3}{4}},$$

$$C_d(x) = \frac{\tau(x)}{\rho U_c^2} = \frac{f''(0)}{c(x) f'^2_{\max}} = \frac{f''(0)}{4(\frac{1}{4}Gr_m)^{\frac{1}{4}} f'^2_{\max}}.$$

The two transport parameters, Nu_x and $C_d(x)$ depend on Prandtl number through $\phi'(0)$ and $f''(0)$. These values are plotted in figure 2 for $4g\beta/mc_p = 0$.

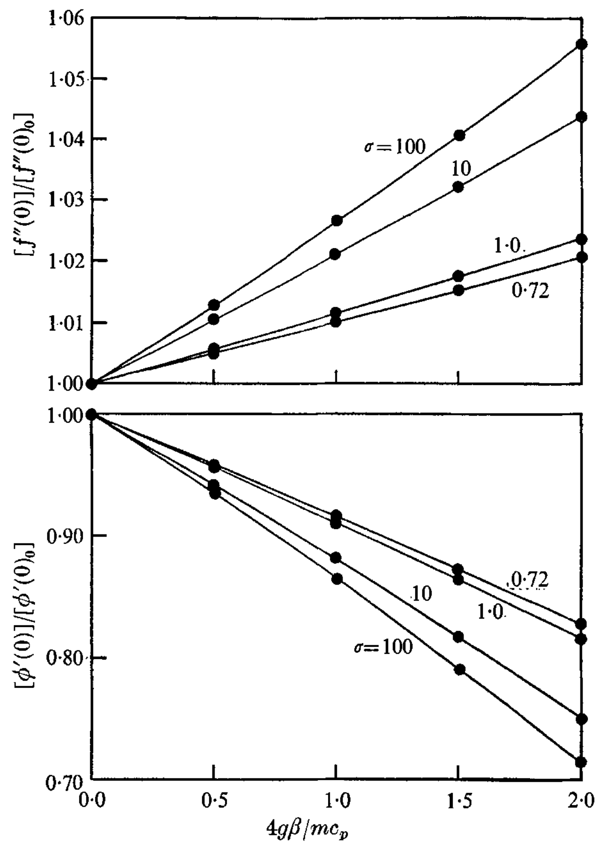

FIGURE 4. Effect of viscous dissipation on heat transfer and viscous drag.

Average transport parameters, $Nu = hL/k$ and $C_d = D/\rho L U_{c,L}^2$ where h is the average heat flux divided by the average temperature difference $\overline{\Delta t}$ and D is the integral of $\tau(x)$ over the interval $x = 0$ to L, are:

$$Nu = \frac{2\sqrt{2}}{5}[-\phi'(0)]\frac{\sqrt{(mL)}\,[e^{\frac{5}{4}mL}-1]}{[e^{mL}-1]^{\frac{5}{4}}}(Gr_{L,\overline{\Delta t}})^{\frac{1}{4}},$$

$$C_d = \frac{4}{3\sqrt{(mL)}}(e^{\frac{3}{4}mL}-1)\,(e^{mL}-1)^{\frac{1}{4}}\,(Gr_{L,\overline{\Delta t}})^{\frac{1}{4}},$$

where $$Gr_L = g\beta L^3 \overline{\Delta t}/\nu^2.$$

The effect of viscous dissipation as a distributed energy source in the convection region is shown in figure 3. Velocity and temperature distributions are both shown, for a Prandtl number of 100, for $4g\beta/mc_p$ of zero and 2·0. The effect is an increase in convection velocity and a reduction of temperature gradient at the surface, and therefore, in heat transfer rate, as $4g\beta/mc_p$ increases.

These effects are more clearly seen in figure 4, where the heat transfer and drag parameters $\phi'(0)$ and $f''(0)$ are shown for various Prandtl numbers over a range of $4g\beta/mc_p$. The values are normalized by their respective values at zero level of viscous dissipation, denoted as $\phi'(0)_0$ and $f''(0)_0$. The effect on heat transfer is seen to be proportionately much greater than that on drag. The effect increases with Prandtl number and may be much larger for the much higher values associated with even more viscous liquid. Calculated numerical values of these parameters are tabulated.

	$4g\beta/mC_p$				
σ	0·0	0·5	1·0	1·5	2·0
0·72	0·74114, 0·63414	0·71040, 0·63734	0·67894, 0·64060	0·64673, 0·64392	0·61376, 0·64729
1·0	0·82354, 0·60208	0·78736, 0·60557	0·75019, 0·60912	0·71201, 0·61274	0·67279, 0·61644
10·0	1·61719, 0·38853	1·52368, 0·39256	1·42552, 0·39675	1·32239, 0·40111	1·21391, 0·40564
100	2·980, 0·231	2·786, 0·234	2·579, 0·237	2·360, 0·240	2·126, 0·244

TABLE 1. Heat transfer and drag parameters, $\phi'(0), f''(0)$

Conclusions

The foregoing results are the first exact calculation of the viscous dissipation effect in a multi-dimensional natural convection flow. This similarity case makes the effect clear and will permit its calculation even for processes in which dissipation is the dominant process.

It is true for these results, as for those published earlier, that these effects will be important only in what are now considered quite extreme physical processes. However, moderate acceleration, i.e. conditions of say $10^4 g$, may cause a 20 % reduction in heat transfer in gases and in common liquids. Higher Prandtl number fluids show a much higher effect in terms of $4g\beta/mc_p$ and the results for much higher values of σ would be interesting. The double boundary-layer idea should again be applicable and an asymptotic behaviour (in σ) should be found.

It is emphasized that appreciable viscous dissipation effects do not preclude laminar flow. Stability and transition are in terms of Grashof number limits. The dissipation parameter is entirely independent of Grashof number. However, the viscous dissipation energy source results in an additional term in the stability equations. Its effect is now unknown.

The results shown in figure 4 extend to $4g\beta/mc_p = 2·0$. The behaviour of the heat transfer parameter $-\phi'(0)$ beyond this limit would be very interesting. By a value of perhaps 10 the dissipation effect should dominate the other physical effects. The curves go sharply towards zero, do they approach different non-zero asymptotes?

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Appendix A

The exponential boundary conditions result in similar solutions of boundary-layer equations in many flow circumstances. However, it is necessary to be concerned about physical implications and characteristics. There are particular peculiarities of the exponential surface temperature variation in natural convection. These imply limitations of the permissible interpretation and applicability of the results.

Such analysis, as the foregoing, proceeds as though flow is considered over a flat surface at $y = 0$ extending from $x = 0$ to infinity. The boundary region thickness $y = \delta$ (at η_δ) is as follows:

$$\begin{aligned}\delta &= \eta_\delta / b(x) \\ &= \eta_\delta x/(\tfrac{1}{4}Gr_x)^{\frac{1}{4}} = \eta_\delta \left(\frac{4\nu^2}{gBN}\right)^{\frac{1}{4}} x^{\frac{1}{4}(1-m)} \\ &= \eta_\delta/(\tfrac{1}{4}Gr_m)^{\frac{1}{4}} = \eta_\delta \left(\frac{4\nu^2 m^3}{g\beta M}\right)^{\frac{1}{4}} e^{-\frac{1}{4}mx},\end{aligned}$$

for the power law and exponential case respectively. It is seen that for all permissible power law cases ($n > -0{\cdot}6$) for a flat surface δ goes to zero at $x = 0$ for $n < 1$. Therefore, no momentum or energy flows on to the surface at the leading edge for such cases. However, all exponential cases have a non-zero value of δ at $x = 0$, there is no singularity. Since the similarity solution applies at $x = 0$, a non-zero momentum $\dot{M}$ and energy flow $\dot{Q}$ are implied at the leading edge. This is a common problem in the application of boundary-layer theory and, in this case, causes a fundamental inaccuracy in the use of these results for a plate lying at $x \geqslant 0$.

The inaccuracy is estimated here by finding for what length of the plate (to $x = L$) the leading edge contribution of $\dot{M}$ and $\dot{Q}$ is important, compared to the local values. The $\dot{M}$ and $\dot{Q}$ at any x are

$$\begin{aligned}\dot{M}(x) &= \int_0^\infty \rho u^2\,dy = \rho\nu^2 c^2 b \int_0^\infty (f')^2\,d\eta \\ &= 16\rho\nu^2 m \left(\frac{g\beta M}{\nu^2 m^3}\right)^{\frac{3}{4}} e^{\frac{3}{4}mx} \int_0^\infty (f')^2\,d\eta,\end{aligned}$$

$$\begin{aligned}\dot{Q}(x) &= \int_{-\infty}^\infty \rho u c_p (t - t_\infty)\,dy = 2\rho c_p \nu c d \int_0^\infty \phi f'\,d\eta \\ &= \frac{8\rho c_p \nu M}{\sqrt{2}} \left(\frac{g\beta M}{\nu^2 m^3}\right)^{\frac{1}{4}} e^{\frac{5}{4}mx} \int_0^\infty \phi f'\,d\eta.\end{aligned}$$

The ratio of the values of $\dot{M}$ and $\dot{Q}$ at $x = L$ to their values at the leading edge are

$$\dot{M}(L)/\dot{M}(0) = e^{\frac{3}{4}mL},$$
$$\dot{Q}(L)/\dot{Q}(0) = e^{\frac{5}{4}mL}.$$

Clearly the similarity solution is reasonable for the portion of the surface from $x = 0$ to L only if these are large quantities, i.e. if mL is considerably larger than 1.

Another way of interpreting this leading edge flow is as though $x = 0$ is located on a doubly infinite surface ($-\infty$ to $+\infty$) with a surface temperature Me^{mx}. Since C is positive this is an exponential growth for $x > 0$ and decay for $x < 0$. The leading edge convection (at $x = 0$) calculated above is the convection produced over the whole of the surface from $-\infty$ to 0. The applicability of the similarity solution to a surface lying only above $x = 0$ depends upon whether the total contribution from $-\infty$ to 0 is negligible compared to that between 0 and L. Clearly large m, fast decay, and large L are the conditions.

Appendix B

It was shown earlier that there is no similarity for the power law case when viscous dissipation is included. The x dependence appearing in the viscous dissipation term suggests a perturbation about the non-dissipation flow. This was done by Gebhart (1962) for various Prandtl numbers, for the isothermal and uniform flux boundary conditions. The analysis is given here for any power law surface temperature distribution.

Assuming a similarity variable and stream function of the form

$$\eta = yb(x),$$
$$\psi(x,y) = c(x)\,[f_0(\eta) + \epsilon(x) f_1(\eta) + \epsilon^2(x) f_2(\eta) + \ldots],$$

and
$$\phi(x,y) = \phi_0(\eta) + \epsilon(x)\,\phi_1(\eta) + \epsilon^2(x)\,\phi_2(\eta) + \ldots.$$

The functions $b(x)$ and $c(x)$ are chosen corresponding to the power law case, equation (11). The zeroth-order terms in ϵ are to be the non-dissipation similarity solution. This occurs if ϵ is chosen as

$$\epsilon(x) = 4\frac{g\beta x}{c_p}.$$

For f_0 and ϕ_0 we have

$$f_0''' + \phi_0 + (n+3) f_0 f_0'' - (2n+2)(f_0')^2 = 0,$$
$$\phi_0'' + \sigma[(n+3) f_0 \phi_0' - 4n f_0' \phi_0] = 0.$$

The equations from first-order terms in ϵ, i.e. for f_1 and ϕ_1, are

$$f_1''' + (7+n) f_0'' f_1 + (3+n) f_0 f_1'' - (8+4n) f_0' f_1' + \phi_1 = 0,$$
$$\phi_1'' + \sigma[(\eta - n) f_1 \phi_0' + (3+n) f_0 \phi_1' - 4(1+n) f_0' \phi_1 - 4_n f_1' \phi_0 + (f_0'')^2] = 0.$$

Additional equations may be written for higher-order terms; they amount to two coupled relations at each level.

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